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The submerged sphere as an absorber of wave power

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A submerged sphere is considered to be absorbing power from an incident wave through an integrated mooring and power take-off system. It is shown that the power absorbed (as characterized by the absorption length) depends on the hydrodynamic properties of the sphere; in particular on the added-mass and damping coefficients. These coefficients are determined and the results used to study the power absorption properties of the sphere. Curves are given showing the variation of the absorption length with wavenumber, for differing depths of submergence.

1. Introduction

In a paper in the *Journal of Fluid Mechanics* in 1976, Evans proposed a theory for the absorption of wave power by oscillating bodies. In § 7 of his paper he considered the power absorption properties of a three-dimensional body with a vertical axis of symmetry. For such a body, constrained to move in only one mode, he proved the remarkable result that

$$l_{\max} = \epsilon_i \lambda (2\pi)^{-1},$$

where $l_{\max}$ is the maximum absorption length‡ and λ the wavelength of the incident wave. Here $\epsilon_i = 2$ for $i = 1, 3$ and $\epsilon_i = 1$ for $i = 2$ where $i = 1, 2, 3$ refer to sway, heave and roll motions respectively. This result shows that the maximum power absorbed is independent of the size of the body and depends only on the wavelength of the incident wave and the mode of motion of the body.

Evans applied his results to the particular case of a floating sphere, moving in heave. However, he was unable to take full advantage of the above result, in the following sense. Non-dimensionalizing the maximum absorption length with respect to the diameter, $2a$, of the sphere gives

$$l_{\max}/2a = (2Ka)^{-1},$$

where $K = 2\pi\lambda^{-1}$ is the wavenumber. For a good absorber $l_{\max}/2a$ must be greater than one; that is the power absorbed must be equivalent to the power in an incident wave whose crest length is greater than the diameter of the sphere. This clearly requires Ka to be less than a half and thus the sphere must be tuned to non-dimensional wavenumbers $Ka < \frac{1}{2}$. For a floating sphere, Evans found that the natural buoyancy restoring force made it impossible to achieve this tuning (see Evans 1976, § 7, for details).

In this paper a submerged sphere is considered to be absorbing power from an

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‡ Absorption length = power absorbed by the body divided by the total power in an incident wave of unit frontage (see Evans 1976, § 7).

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incident wave. This enables the difficulties caused by the natural buoyancy restoring force, which acts on the floating sphere, to be overcome because a submerged sphere does not experience such a force acting upon it. An attempt is also made to model a realistic power take-off system for the sphere. The idea for this power take-off system is taken from Evans, Davis & Srokosz (1978), and is as follows. The sphere is moored by three neutrally-buoyant cables which reach down to housings situated on the sea bed. Each cable is wrapped round a spring-loaded cable drum at its lower end. The sphere is given a slight positive buoyancy so that the cables are in tension when the sea is calm. In waves the sphere is forced to move and the cables wind onto and unwind from the cable drums in response to the motion of the sphere. Throughout the motion power is absorbed by power take-off mechanisms which are attached to the cable drums and which convert the motion into usable power. The system is modelled in this paper by linear dampers and springs which have a resistance to motion proportional to the rate of change of extension and the extension of the cables respectively.

In order to calculate the absorption length for the submerged sphere it is necessary to consider the equations of motion of the sphere when it is moving in response to an incident wave. The mooring and power take-off system allows the sphere to move in more than one mode. This results in an increased absorption length, as compared to the case when the sphere is constrained to move in only one mode. It is shown that the absorption length depends on the hydrodynamic properties of the sphere; in particular on the added-mass and damping coefficients. These calculations are presented in part (*a*), §§ 3–6, of this paper. In § 6 curves for the absorption length of the submerged sphere are presented and discussed. Results are also given for a submerged sphere moored by only one cable. These are derived as a special case of the three cable system.

In part (*b*), §§ 7–11, of this paper the radiation problems for a heaving, surging and swaying submerged sphere are solved. The radiation problem is one in which there is no wave incident upon the sphere and the sphere is forced to oscillate in a given mode. The solutions to the radiation problems include the values of the added-mass and damping coefficients necessary to calculate the absorption length in part (*a*) of this paper. The radiation problems are solved by using the multipole potentials derived by Thorne (1953); following the method used by Ursell (1950) to solve the problem of the scattering of an incident wave by a submerged circular cylinder. Curves of the added-mass and damping coefficients are presented and discussed in § 11.

2. Formulation

Cartesian co-ordinates (x, y, z) are chosen such that $y = 0$ is the undisturbed free surface, with y measured vertically downwards. The sphere has radius a and its centre is at $(0, h, 0)$ (see figure 1) with $a/h < 1$. Consider the motion of the submerged sphere in response to an incident wave. The usual assumptions of linearized water wave theory allow the introduction of a velocity potential $\Phi(x, y, z; t)$ satisfying

$$\nabla^2\Phi = 0 \quad \text{in the fluid,} \tag{2.1}$$

the linearized free-surface condition

$$\frac{\partial^2\Phi}{\partial t^2} - g\frac{\partial\Phi}{\partial y} = 0 \quad \text{on} \quad y = 0, \tag{2.2}$$

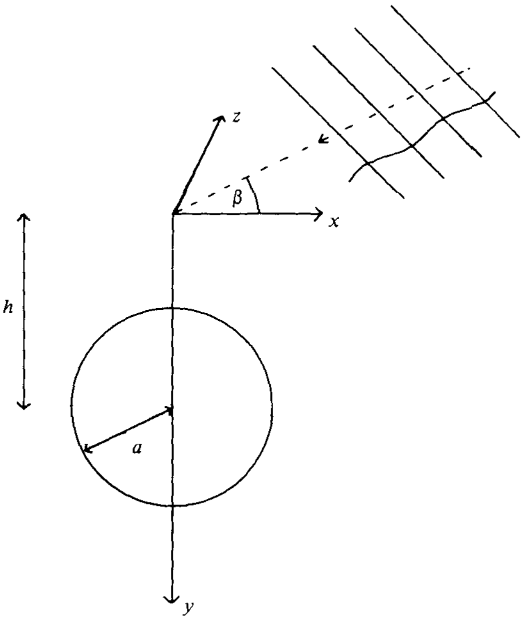

FIGURE 1. Co-ordinate system for the submerged sphere.

and

$$\nabla\Phi \to 0 \quad \text{as} \quad y \to \infty. \tag{2.3}$$

It is assumed that a small amplitude sinusoidal wave train, of frequency ω, is incident upon the sphere from a direction making an angle β with the x axis. Under the constraints of the mooring and power take-off system the sphere moves in response to the incident wave. Its motion may be represented by a linear combination of motions in surge, heave and sway modes, that is, by a linear combination of motions parallel to the x, y, z axes respectively.

Let $\zeta_j(t)$ be the displacement of the sphere, in the jth mode from its equilibrium position. Here $j = 1$ relates to surge, $j = 2$ to heave and $j = 3$ to sway. The linearized condition of equal normal velocity of sphere and fluid applied on the equilibrium position of the sphere is

$$\frac{\partial\Phi}{\partial n} = \sum_{j=1}^{3} \dot{\zeta}_j(t)\, n_j, \tag{2.4}$$

for (x, y, z) on S, the surface of the sphere; where $\mathbf{n} = (n_1, n_2, n_3)$ is the normal vector from the sphere into the fluid at the point (x, y, z).

It is possible to eliminate the harmonic time dependence by writing

$$\Phi(x, y, z; t) = \operatorname{Re}\{\phi(x, y, z)\, e^{i\omega t}\}, \tag{2.5}$$

and

$$\zeta_j(t) = \operatorname{Re}\{\xi_j\, e^{i\omega t}\}. \tag{2.6}$$

Now the complex-valued time-independent potential $\phi(x, y, z)$ may be written as

$$\phi(x, y, z) = gA\omega^{-1}[\phi_I + \phi_D] + i\omega \sum_{j=1}^{3} \xi_j\, \phi_j, \tag{2.7}$$

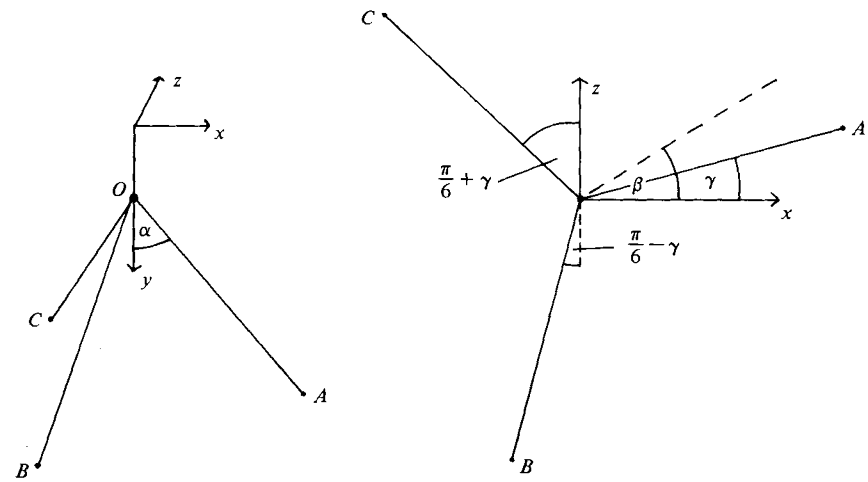

FIGURE 2. Schematic layout of the three cable mooring and power take-off system.

where A is the amplitude of the incident wave and ϕ_I is given by

$$\phi_I = \exp\{iK(x\cos\beta + z\sin\beta) - Ky\}, \tag{2.8}$$

(here $K = \omega^2/g$). The complex potential ϕ_D is the diffracted wave produced if the sphere is held fixed in the presence of the incident wave ϕ_I. The complex potential ϕ_j is the solution to the radiation problem in which the normal velocity $\mathrm{Re}\{n_j e^{i\omega t}\}$ is prescribed on the sphere, corresponding to oscillations in one of the three modes. These radiation problems are considered in part (*b*) of this paper.

Condition (2.4) is satisfied if

$$\frac{\partial\phi_D}{\partial n} = -\frac{\partial\phi_I}{\partial n}, \quad \frac{\partial\phi_j}{\partial n} = n_j \quad \text{on} \quad S. \tag{2.9}$$

(*a*) *The submerged sphere as a wave-power absorber*

3. Equations of motion of the sphere

In order to model the power take-off and mooring system of the sphere consider the situation shown in figure 2. The sphere is at the point 0 and is moored by three cables of length L; each inclined at an angle α to the vertical and situated symmetrically around the sphere. Here a symmetrical mooring and power take-off systems is chosen in order to ensure that the power absorption characteristics of the sphere are independent of the direction of incidence of the incident wave. In calm seas the system is in equilibrium. Thus the tension in the cables, T_0, is given by

$$3T_0\cos\alpha + mg = \tfrac{4}{3}\pi a^3\rho g, \tag{3.1}$$

where m is the mass of the sphere and ρ the density of water. For the cables to be in tension ($T_0 > 0$) the sphere must have positive buoyancy (i.e. $m < \frac{4}{3}\pi a^3\rho$). The value $m = 0{\cdot}8(\frac{4}{3}\pi a^3\rho)$ is used throughout the calculations in this paper.

When the sphere moves in response to the incident wave it is displaced from its equilibrium position $(0, h, 0)$ to $(\zeta_1(t), h+\zeta_2(t), \zeta_3(t))$. Each of the three cables OA, OB,

OC must therefore change in length. The power take-off system is modelled by linear springs and dampers which have a resistance to motion proportional to the extension and the rate of change of extension of the cables, respectively. In order to preserve symmetry it is assumed that the spring and damper rates are the same for all three cables. Thus the tension in the cables OA, OB, OC, when the sphere is in motion is given by

$$\left.\begin{aligned} T_{OA} &= T_0 + k\,\delta A + d\,\delta\dot{A}, \\ T_{OB} &= T_0 + k\,\delta B + d\,\delta\dot{B}, \\ T_{OC} &= T_0 + k\,\delta C + d\,\delta\dot{C}, \end{aligned}\right\} \tag{3.2}$$

where k, d are the spring and damper rates; and δA, δB, δC are the extensions of OA, OB, OC respectively.

A simple calculation shows that

$$\begin{aligned} \delta A &= [(L\sin\alpha\cos\gamma - \zeta_1)^2 + (L\cos\alpha - \zeta_2)^2 + (L\sin\alpha\sin\gamma - \zeta_3)^2]^{\frac{1}{2}} - L, \\ \delta B &= [(L\sin\alpha\sin(\pi/6-\gamma) + \zeta_1)^2 + (L\cos\alpha - \zeta_2)^2 + (L\sin\alpha\cos(\pi/6-\gamma) + \zeta_3)^2]^{\frac{1}{2}} - L, \\ \delta C &= [(L\sin\alpha\sin(\pi/6+\gamma) + \zeta_1)^2 + (L\cos\alpha - \zeta_2)^2 + (L\sin\alpha\cos(\pi/6+\gamma) - \zeta_3)^2]^{\frac{1}{2}} - L. \end{aligned}$$

Under the assumptions of linearized theory the displacements of the sphere from its equilibrium position will be small compared to the length of the cables; that is

$$|\zeta_j(t)/L| \ll 1 \quad \text{for} \quad j = 1, 2, 3.$$

The above equations then give

$$\left.\begin{aligned} \delta A &\simeq -\zeta_1\sin\alpha\cos\gamma - \zeta_2\cos\alpha - \zeta_3\sin\alpha\cos\gamma, \\ \delta B &\simeq \zeta_1\sin\alpha\sin(\pi/6-\gamma) - \zeta_2\cos\alpha + \zeta_3\sin\alpha\cos(\pi/6-\gamma), \\ \delta C &\simeq \zeta_1\sin\alpha\sin(\pi/6+\gamma) - \zeta_2\cos\alpha - \zeta_3\sin\alpha\cos(\pi/6+\gamma). \end{aligned}\right\} \tag{3.3}$$

Note that in order to derive these results the lengths of the cables have been assumed to be finite. Equation (2.3), however, indicates that the sphere is considered to be in infinitely deep water. Therefore to be strictly consistent the length L of the cables should be allowed to tend to infinity. This is not necessary as for this linear theory the extensions of the cables do not depend on L (see (3.3)). In fact allowing L to tend to infinity ensures that the condition $|\zeta_j(t)/L| \ll 1$ is always satisfied for finite displacements $\zeta_j(t)$.

Without loss of generality, assume $\beta = 0$ (see figure 1); that is the waves are incident upon the sphere from $x = +\infty$. Thus, by symmetry, there are no hydrodynamic forces acting on the sphere in the z direction owing to the incident wave. Resolving the motion into components parallel to the x, y, z axes gives the following equations of motion, after the use of (3.1) and (3.2),

$$\begin{aligned} m\ddot{\zeta}_1 = {} & k[\delta A\sin\alpha\cos\gamma - \delta B\sin\alpha\sin(\pi/6-\gamma) - \delta C\sin\alpha\sin(\pi/6+\gamma)] + \\ & d[\delta\dot{A}\sin\alpha\cos\gamma - \delta\dot{B}\sin\alpha\sin(\pi/6-\gamma) - \delta\dot{C}\sin\alpha\sin(\pi/6+\gamma)] + \\ & F_{e1} + F_{r1}, \\ m\ddot{\zeta}_2 = {} & k[\delta A\cos\alpha + \delta B\cos\alpha + \delta C\cos\alpha] + d[\delta\dot{A}\cos\alpha + \delta\dot{B}\cos\alpha + \\ & \delta\dot{C}\cos\alpha] + F_{e2} + F_{r2}, \end{aligned}$$

$$m\ddot{\xi}_3 = k[\delta A \sin\alpha \sin\gamma - \delta B \sin\alpha \cos(\pi/6-\gamma) + \delta C \sin\alpha \cos(\pi/6+\gamma)] + d[\delta\dot{A} \sin\alpha \sin\gamma - \delta\dot{B} \sin\alpha \cos(\pi/6-\gamma) + \delta\dot{C} \sin\alpha \cos(\pi/6+\gamma)] + F_{r3}.$$

Here F_{ej} is the exciting force acting on the sphere in the jth mode, due to the incident wave, when the sphere is held fixed. F_{rj} is the radiation force acting on the sphere in the jth mode owing to its own motion, in the absence of the incident wave. Note that there is no exciting force due to the incident wave acting parallel to the z-axis by symmetry. In general, for a body moving in surge, heave and sway, F_{rj} may be written as

$$F_{rj} = -\sum_{k=1}^{3} \{a_{jk}\ddot{\xi}_k + b_{jk}\dot{\xi}_k\} \quad \text{for} \quad j = 1, 2, 3, \tag{3.4}$$

where a_{jk}, b_{jk} are the added-mass and damping coefficients. For a sphere, $a_{jk} = 0$, $b_{jk} = 0$ whenever $j \neq k$ by symmetry; that is motion in one mode does not produce hydrodynamic forces acting in other modes. Note also that F_{ej} may be written as

$$F_{ej} = \mathrm{Re}\{X_j e^{i\omega t}\}. \tag{3.5}$$

For further details of the form of F_{rj} and F_{ej} see Newman (1976).

Using (2.6) and (3.3)–(3.5), the equations of motions may be written, after some tidying up, as

$$[-(m+a_{11})\omega^2 + i\omega b_{11}]\xi_1 = X_1 - \tfrac{3}{2}k\sin^2\alpha\xi_1 - \tfrac{3}{2}i\omega d\sin^2\alpha\xi_1, \tag{3.6}$$

$$[-(m+a_{22})\omega^2 + i\omega b_{22}]\xi_2 = X_2 - 3k\cos^2\alpha\xi_2 - 3i\omega d\cos^2\alpha\xi_2, \tag{3.7}$$

$$[-(m+a_{33})\omega^2 + i\omega b_{33}]\xi_3 = -\tfrac{3}{2}k\sin^2\alpha\xi_3 - \tfrac{3}{2}i\omega d\sin^2\alpha\xi_3. \tag{3.8}$$

As may have been deduced from the symmetry of the system, the motions in three mutually perpendicular directions are independent of one another, except for the dependence of all three equations on k, d and α. Equation (3.8) shows that $\xi_3 = 0$; that is, there is no motion in the horizontal direction perpendicular to the line of incidence of the incident wave. In the next section these results are used to consider the power absorption properties of the submerged sphere.

4. Power absorption by the submerged sphere

The power P absorbed by the sphere is the mean rate at which work is being done on the sphere by the fluid. That is

$$P = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \sum_{j=1}^{3} [F_{ej} + F_{rj}]\,\dot{\xi}_j(t)\,dt, \\ = \tfrac{3}{4}\omega^2 d[\sin^2\alpha|\xi_1|^2 + 2\cos^2\alpha|\xi_2|^2],$$

using $\xi_3 = 0$, (3.4)–(3.7) and (2.6). The absorption length l is now defined as the power absorbed divided by the mean energy-flux per unit length across a vertical plane normal to the wave direction. Thus l is given by

$$l = P/[(4\omega)^{-1}\rho g^2|A|^2], \\ = \frac{3\omega^3 d}{\rho g^2}\left[\sin^2\alpha\left|\frac{\xi_1}{A}\right|^2 + 2\cos^2\alpha\left|\frac{\xi_2}{A}\right|^2\right]. \tag{4.1}$$

Newman [1962; equations (31)–(33)] proved that relations exist betwen the exciting forces F_{ej} and the damping coefficients b_{jj}. Those may be written, on use of (3.5), as

$$|X_j|^2 = \epsilon_j\, 2\rho g^3 \omega^{-3} |A|^2 b_{jj} \quad \text{for} \quad j = 1, 2, 3, \tag{4.2}$$

where

$$\epsilon_j = \begin{cases} 2 & \text{for} \quad j = 1, 3, \\ 1 & \text{for} \quad j = 2. \end{cases}$$

It follows, from (3.6), (3.7), (4.1) and (4.2), that the absorption length is given by

$$l = K^{-1} \left\{ \frac{48\omega^2 b_{11} d \sin^2\alpha}{[3k \sin^2\alpha - 2(m + a_{11})\,\omega^2]^2 + \omega^2 [3d \sin^2\alpha + 2b_{11}]^2} + \frac{12\omega^2 b_{22}\, d \cos^2\alpha}{[3k \cos^2\alpha - (m + a_{22})\,\omega^2]^2 + \omega^2 [3d \cos^2\alpha + b_{22}]^2} \right\}. \tag{4.3}$$

Alternatively, write l as

$$l = K^{-1}(l_1 + l_2), \tag{4.4}$$

where

$$l_1 = 2\left\{1 - \frac{[3k \sin^2\alpha - 2(m + a_{11})\,\omega^2]^2 + \omega^2 [3d \sin^2\alpha - 2b_{11}]^2}{[3k \sin^2\alpha - 2(m + a_{11})\,\omega^2]^2 + \omega^2 [3d \sin^2\alpha + 2b_{11}]^2}\right\}, \tag{4.5}$$

and

$$l_2 = \left\{1 - \frac{[3k \cos^2\alpha - (m + a_{22})\,\omega^2]^2 + \omega^2 [3d \cos^2\alpha - b_{22}]^2}{[3k \cos^2\alpha - (m + a_{22})\,\omega^2]^2 + \omega^2 [3d \cos^2\alpha + b_{22}]^2}\right\}. \tag{4.6}$$

Note that the absorption length l depends on the values of the added mass and damping coefficients for the submerged sphere. These coefficients are determined in part (*b*) of this paper and are used in calculating the absorption length.

In order to find the maximum absorption length, l must be maximized as a function of k, d for given ω and α. Although this appears straightforward it is in fact difficult to maximise l analytically; therefore a numerical maximisation procedure was carried out by computer. Details of the maximization procedure are given in §6. From (3.6), (3.7) and (4.2) it is possible to calculate the values $|\xi_1/A|$ and $|\xi_2/A|$, which give the time-independent displacements of the sphere from its equilibrium position. They are non-dimensionalized with respect to the amplitude of the incident wave. Results for the displacements of the sphere and its absorption length are presented and discussed in §6.

When l is written as the sum of two parts, (4.4)–(4.6), it is clear that each part can be maximized separately, for a given ω and α. Thus $l_1 = 2$ when

$$k = 2(m + a_{11})\,\omega^2/3\sin^2\alpha, \quad d = 2b_{11}/3\sin^2\alpha, \tag{4.7}$$

and $l_2 = 1$ when

$$k = (m + a_{22})\,\omega^2/3\cos^2\alpha, \quad d = b_{22}/3\cos^2\alpha. \tag{4.8}$$

In general, k, d cannot be chosen such that (4.7) and (4.8) hold simultaneously. It may therefore be deduced that the maximum absorption length $l_{\max} < 3K^{-1}$. If the power take-off systems in heave and surge were independent, that is, not through the system of cables modelled here, then it would be possible to obtain $l_{\max} = 3K^{-1}$.

5. Special case of a single-cable mooring and power take-off system

If α is set to zero in the above analysis then the following results are obtained

$$\left|\frac{\xi_1}{A}\right|^2 = \frac{4\rho g^3\omega^{-3}b_{11}}{[(m+a_{11})^2\omega^4+b_{11}^2\,\omega^2]}\Bigg\}, \tag{5.1}$$

$$\left|\frac{\xi_2}{A}\right|^2 = \frac{2\rho g^3\omega^{-3}b_{22}}{[k_1-(m+a_{22})\,\omega^2]^2+\omega^2(b_{22}+d_1)^2}, \tag{5.2}$$

and

$$l = K^{-1}\left\{1-\frac{[k_1-(m+a_{22})\,\omega^2]^2+\omega^2(b_{22}-d_1)^2}{[k_1-(m+a_{22})\,\omega^2]^2+\omega^2(b_{22}+d_1)^2}\right\}, \tag{5.3}$$

where $k_1 = 3k$ and $d_1 = 3d$. These results clearly apply to the situation in which the power take-off and mooring for the sphere are provided by one vertical cable. Now l may be maximized as a function of k_1 and d_1 to give

$$l_{\max} = K^{-1}, \tag{5.4}$$

when

$$k_1 = (m+a_{22})\,\omega^2, \quad d_1 = b_{22}. \tag{5.5}$$

These results show that power is only extracted from the incident wave through the heaving motion of the sphere. The surge oscillations of the sphere are identical to those of a totally unconstrained sphere moving in response to an incident wave. The maximum value of l is the same as that obtained by Evans (1976) for an axisymmetric body moving only in heave and absorbing energy from an incident wave.

Calculations were carried out for this case of a single cable mooring and power take-off system and also the case of a three cable mooring and power take-off system. The results obtained are presented and discussed in the next section.

6. Results and discussion for the absorption length of the sphere

If k and d are chosen such that $l = l_{\max}$ at $\omega = \omega_0$, for a given α, then $k = k(\omega_0)$, $d = d(\omega_0)$, $l_{\max} = l_{\max}(\omega_0)$ all depend on ω_0. Substituting $k(\omega_0)$, $d(\omega_0)$ into (4.5) (or (5.3)) shows that l depends on ω and ω_0 and has the following properties:

$$\left.\begin{aligned} l = l(\omega,\omega_0) \quad \text{where} \quad & l(\omega_0,\omega_0) = l_{\max}(\omega_0) \\ \text{and} \quad & l(\omega,\omega_0) \leqslant l_{\max}(\omega). \end{aligned}\right\} \tag{6.1}$$

In this case the sphere is said to be 'tuned' to $\omega = \omega_0$. Now, for a given ω_0, $k(\omega_0)$ and $d(\omega_0)$ need to be calculated and then the variation of $l(\omega,\omega_0)$ with ω may be studied.

In all the calculations below the mass of the sphere is taken to be 0·8 times the mass of the displaced fluid, that is

$$m = 0{\cdot}8(\tfrac{4}{3}\pi a^3\rho).$$

This ensures that, when the sphere is in equilibrium, the cables are in tension (see (3.1)). The values of the added-mass and damping coefficients used in the calculations are taken from the results of part (*b*) of this paper.

For the sphere, with the three cable mooring and power take-off system, k, d and $l_{\max}$ were determined for given values of α and ω_0 by a numerical maximization procedure applied to (4.3). This involved setting k and d to some initial value and then the

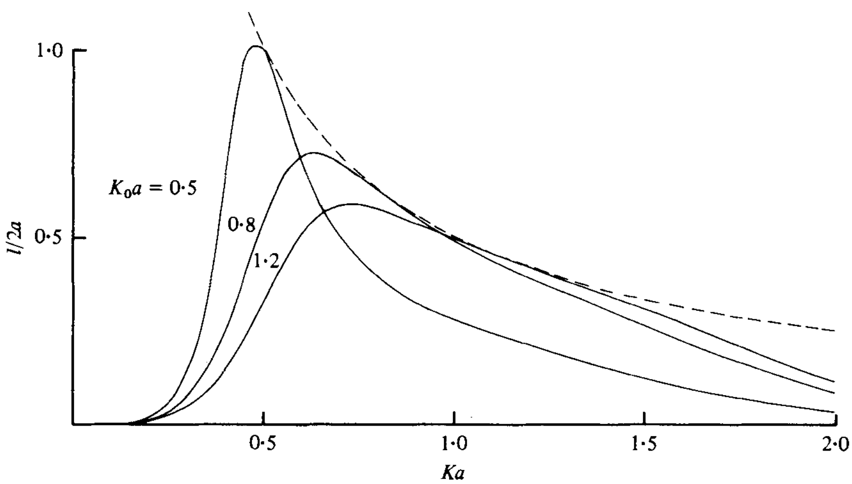

FIGURE 3. Non-dimensional absorption length $l/2a$ *vs.* non-dimensional wavenumber Ka, for the submerged sphere with a single-cable power take-off system, with $a/h = \frac{4}{5}$ for different values of tuned wavenumber $K_0\, a$. The dashed curve represents the non-dimensional maximum absorption length, $l_{max}/2a = \frac{1}{2}(\mathrm{Ka})^{-1}$.

computer procedure found the maximum of the function nearest the initial values of k, d. The procedure also gave the values of k, d for which this maximum was achieved. For each value of ω_0 two initial values of k, d were used; these were the values given by (4.7) and (4.8). This procedure was carried out for various values of the parameters $\alpha, a/h$ and $K_0\, a$ $(= \omega_0^2\, a/g)$ and in each case the maxima obtained from the two different initial values of k, d were compared. Two main features were apparent from the results. Firstly, it was found that, for certain combinations of the parameters, the numerical maximisation procedure gave the same value for $l_{\max}$ and the same final values of k, d irrespective of which of the two initial values for k, d were used. In all cases $l_{\max}$ was such that $2K_0^{-1} < l_{\max} < 3K_0^{-1}$. Secondly, for all other combinations of the parameters considered, the two different initial values of k, d gave two different values of $l_{\max}$ at different final values of k, d. Inspection of the results revealed that if k, d were given initially by (4.8), then $l_{\max}$ was found to be just greater than K_0^{-1}. Similarly, if k, d were given by (4.7) initially, then $l_{\max}$ was found to be just greater than $2K_0^{-1}$. In both cases the final values of k, d, which gave these maxima for l, were found to be close to the initial values. This suggests that, for certain values of the parameters, the system may be tuned to take power mainly out of the heave motion ($l_{\max} \simeq K_0^{-1}$) or mainly out of the surge motion ($l_{\max} \simeq 2K_0^{-1}$). When only one maximum is obtained for l, from the two different initial values of k, d, then power is taken from both heave and surge motions and this results in an increased $l_{\max}$. The results presented here are of the type where l has only one maximum; with one exception which is mentioned below.

For the special case of a single-cable mooring and power take-off system ($\alpha = 0$) it was unnecessary to carry out this numerical maximization procedure for l as

$$l_{\max} = K_0^{-1},$$

at the values of k_1, d_1 given by (5.4), when $\omega = \omega_0$.

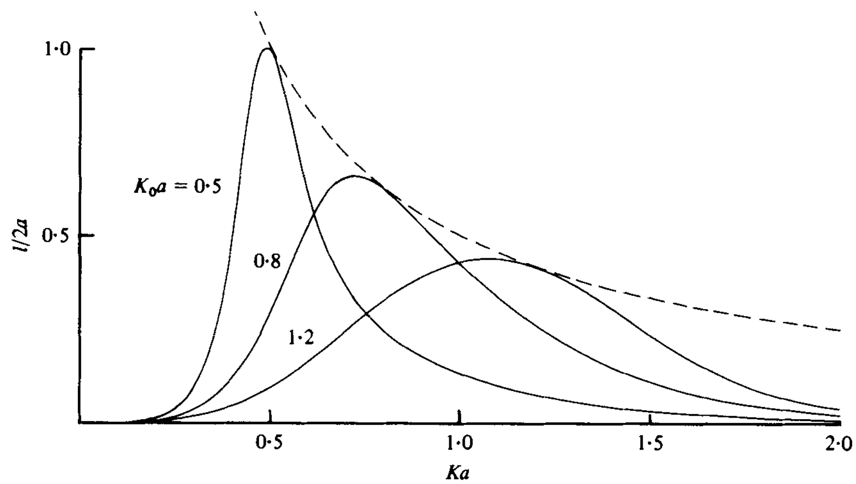

FIGURE 4. Non-dimensional absorption length $l/2a$ *vs*. non-dimensional wave number Ka, for the submerged sphere with a single-cable power take-off system, with $a/h = \frac{5}{8}$ for different values of tuned wavenumber $K_0\,a$. The dashed curve represents the non-dimensional maximum absorption length, $l_{max}/2a$.

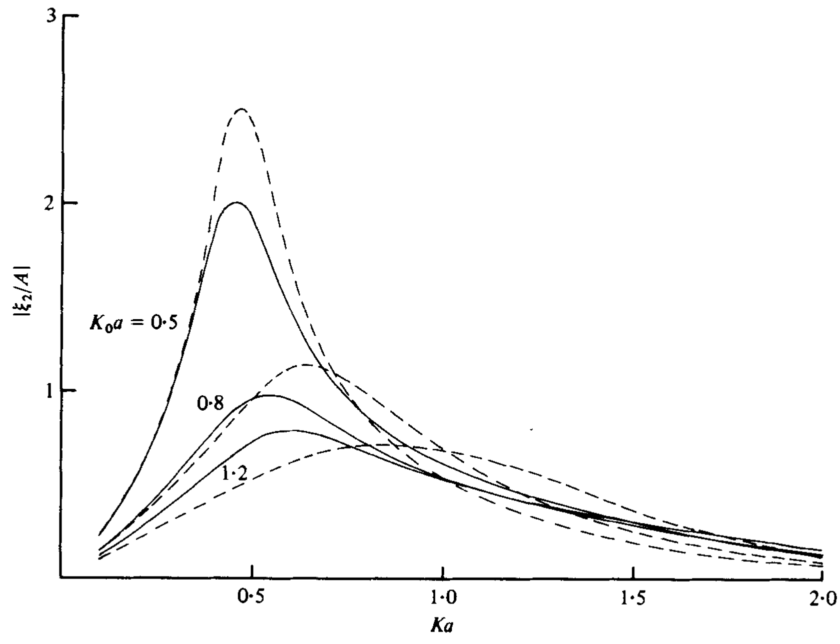

FIGURE 5. Heave-amplitude ratio $|\xi_2/A|$ *vs*. non-dimensional wavenumber Ka, for the submerged sphere with a single-cable power take-off system, for different values of tuned wavenumber $K_0\,a$ ——, $a/h = \frac{1}{8}$; ---, $a/h = \frac{5}{8}$.

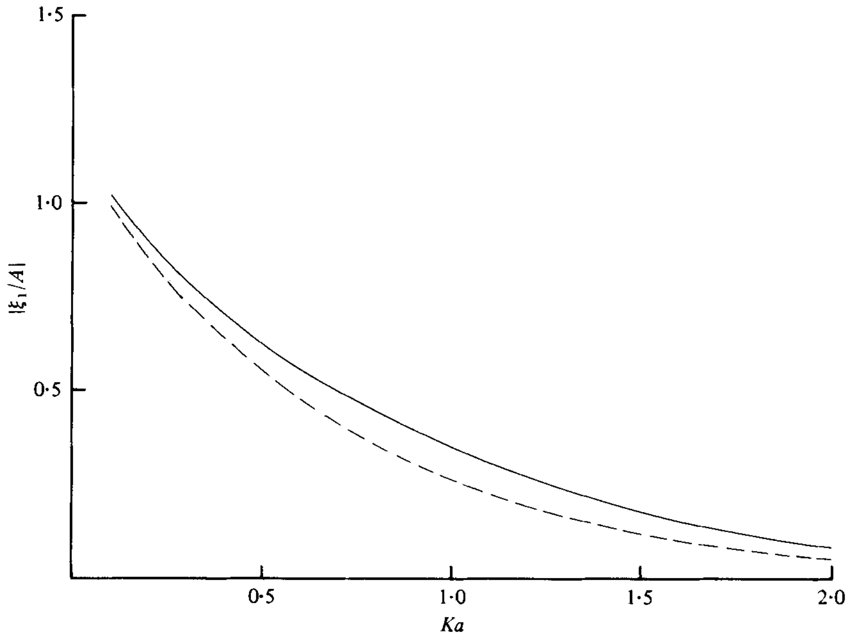

FIGURE 6. Surge amplitude ratio $|\xi_1/A|$ *vs.* non-dimensional wavenumber Ka, for the submerged sphere with a single-cable power take-off system. ——, $a/h = \frac{4}{5}$; ---, $a/h = \frac{2}{3}$.

With the above information it was possible to evaluate l for different values of $K_0 a$, a/h, α and for a range of values of Ka. Values of $|\xi_j/A|$, for $j = 1, 2$, were also calculated and the results give some indication as to the validity of using linearized water wave theory to model this system. Note that the non-dimensional values $l/2a$ and $|\xi_j/A|$ are plotted against Ka, for differing values of α, a/h, $K_0 a$, in each figure.

6.1. *Single-cable mooring and power take-off* (*figures* 3–6)

In figure 3, where $a/h = \frac{4}{5}$, it can be seen that $l/2a$ achieves its maximum value of $(2Ka)^{-1}$ at the relevant values of $K_0 a$. This is also true for the results given in figure 4, where $a/h = \frac{2}{3}$ from both figures it is clear that the maximum value of $l/2a$ on any given curve is not at the tuned non-dimensional wavenumber $K_0 a$. This phenomenon may be explained by considering the curves of $l_{\max}/2a$ shown in the same figures. Although $l/2a$ touches the curve $l_{\max}/2a$ at $K_0 a$, higher values of $l/2a$ can occur; the only restriction being that the curves for $l/2a$ all lie below the curve for $l_{\max}/2a$ and touch that curve at $K_0 a$ (see (6.1)). Note that the deeper submergence of the sphere, in the case $a/h = \frac{2}{3}$ (figure 4), results in a narrowing of the curves for $l/2a$, although the curve for $l_{\max}/2a$ remains unaffected. It is disappointing to note that $l/2a$ is practically always less than one for all the cases considered. This suggests that the single-cable mooring and power take-off system does not give good wave-power absorption properties for the submerged sphere.

It would seem possible to make $l/2a > 1$ by tuning the system to wavenumbers $K_0 a$ with $K_0 a < 0{\cdot}5$. However, figure 5 indicates that as $K_0 a$ decreases $|\xi_2/A|$ increases and thus the assumptions of linearized water wave theory are no longer valid. This means that the results obtained under these assumptions must be suspect. The

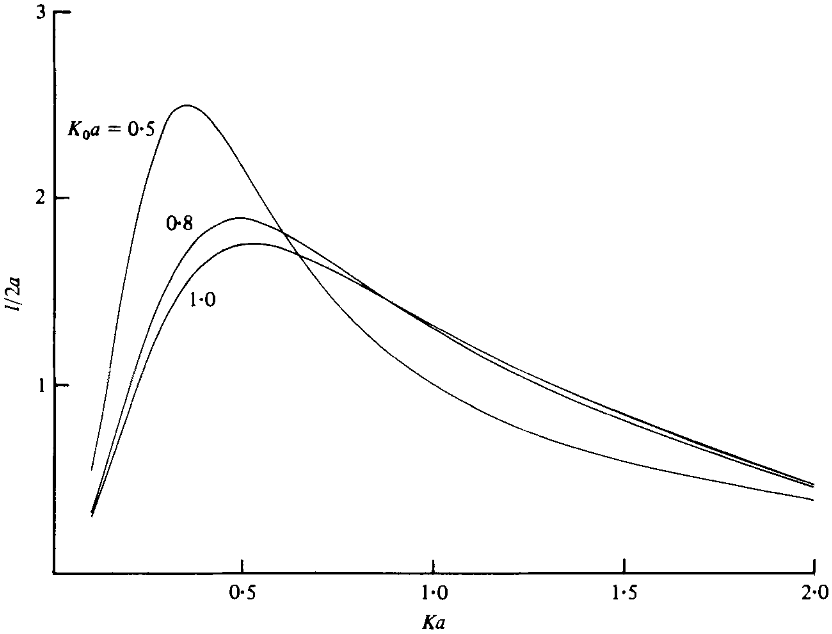

FIGURE 7. Non-dimensional absorption length $l/2a$ *vs.* non-dimensional wavenumber Ka, for the submerged sphere with a three-cable power take-off system, with $\alpha = 60°$, $a/h = \frac{4}{5}$, for different values of tuned wavenumber K_0a.

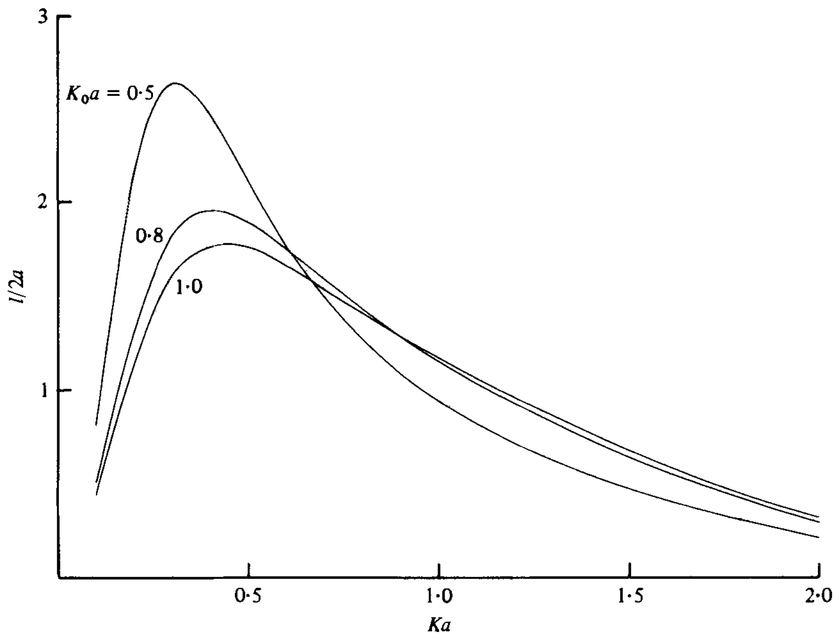

FIGURE 8. Non-dimensional absorption length $l/2a$ *vs.* non-dimensional wavenumber Ka, for the submerged sphere with a three-cable power take-off system, with $\alpha = 60°$, $a/h = \frac{2}{3}$, for different values of tuned wavenumber K_0a.

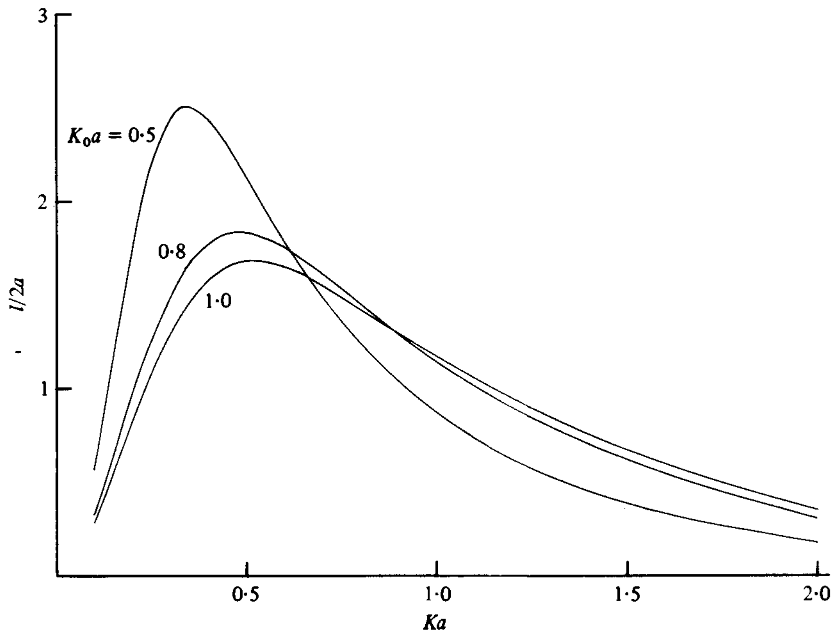

FIGURE 9. Non-dimensional absorption length $l/2a$ *vs.* non-dimensional wavenumber Ka, for the submerged sphere with a three-cable power-take-off system, with $\alpha = 45°$, $a/h = \frac{4}{5}$, for different values of tuned wavenumber $K_0 a$.

cases considered where $K_0 a = 0{\cdot}5$ and $a/h = \frac{4}{5}, \frac{2}{3}$, are probably beyond the limits of the validity of linearized theory. From figure 6 it is clear that the surge motions, which are unaffected by the choice of $K_0 a$, always lie within the bounds of linearised theory for $0{\cdot}1 < Ka < 2$.

6.2. *Three-cable mooring and power take-off* (figures 7–15)

In figures 7–10 curves of $l/2a$ against Ka, for various values of α, a/h and $K_0 a$, are given. All the curves shown, except one, are such that the tuned maximum achieved at $K_0 a$ is of the type where both initial values for k, d gave the same value for l_{max} on use of the numerical maximization procedure (see above). The exception is the curve for $K_0 a = 0.5$ in figure 10 which is obtained by tuning the system to the maximum obtained from the initial values of k, d given by (4.7). In this case the two different initial values of k, d give differing maxima for l, and the system is tuned to the maximum for which most of the power is coming from the surge motions.

The results shown are for the range of parameters α, a/h, $K_0 a$ for which the best results for the absorption length were obtained. As in the case of a single-cable mooring and power take-off system, in figures 7–10 $l/2a$ achieves $l_{max}/2a$ at the tuned wavenumber $K_0 a$ and there are also higher values of $l/2a$ on each curve. The explanation for this is exactly the same as for the single-cable case, only now it is not possible to give an explicit equation for the curve $l_{max}/2a$ beneath which the curves for $l/2a$ lie. Close inspection of the results reveals that, for given a/h, $K_0 a$, if $\alpha = 60°$ there is a slightly greater range of Ka for which $l/2a$ is greater than one than in the case $\alpha = 45°$.

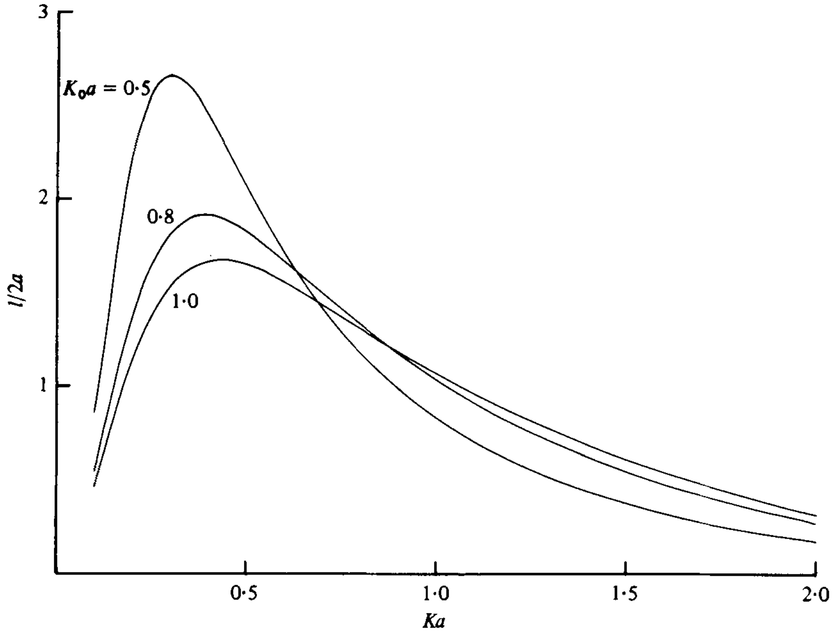

FIGURE 10. Non-dimensional absorption length $l/2a$ *vs.* non-dimensional wavenumber Ka, for the submerged sphere with a three-cable power take-off system, with $\alpha = 45°$, $a/h = \frac{2}{3}$, for different values of tuned wavenumber $K_0 a$.

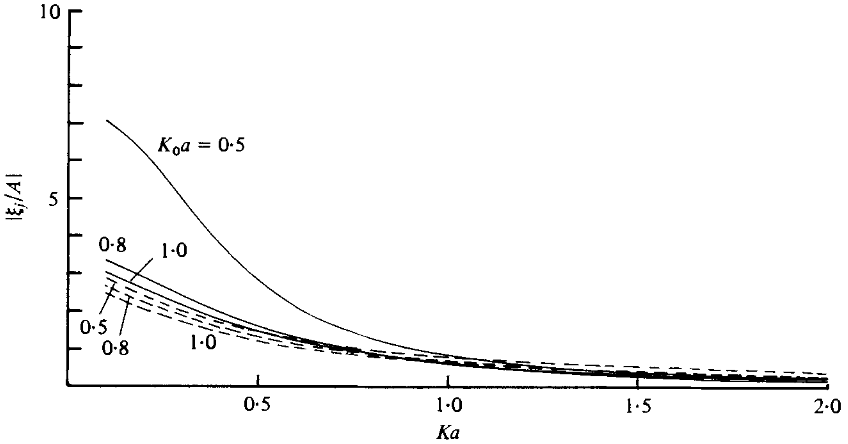

FIGURE 11. Heave and surge amplitude ratios *vs.* non-dimensional wavenumber Ka, for the submerged sphere with a three-cable power take-off system, with $\alpha = 60°$, $a/h = \frac{4}{5}$, for different values of tuned wavenumber $K_0 a$. ——, surge; $|\xi_1/A|$, ------, heave, $|\xi_2/A|$.

It can also be seen that, for given α, K_0, a an increase in depth from $a/h = \frac{4}{5}$ to $a/h = \frac{2}{3}$ results in a narrowing of the range of Ka for which $l/2a$ is greater than one.

While these results look promising a study of figures 11 and 12, which show curves of $|\xi_j/A|$ for $a/h = \frac{4}{5}$, $\frac{2}{3}$ and $\alpha = 60°$, suggest that some of the displacements necessary to achieve good absorption violate the assumption of small oscillations of the sphere. This is especially true when the sphere is tuned to longer waves ($K_0 a = 0{\cdot}5$) and with

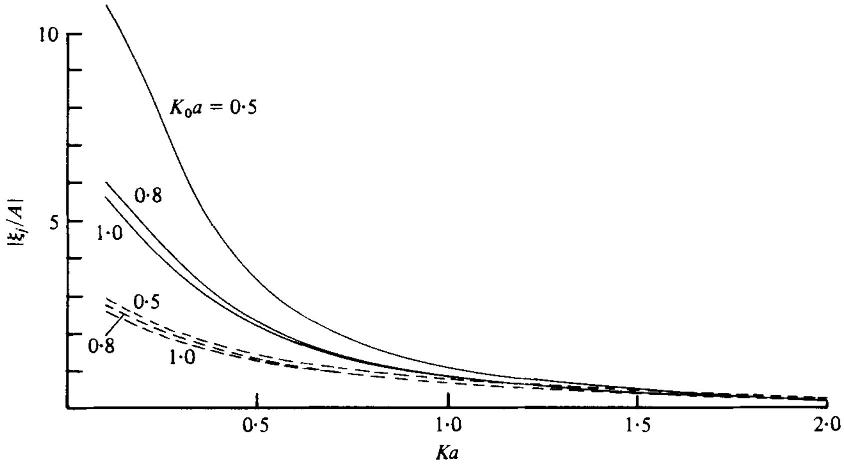

FIGURE 12. Heave and surge amplitude ratios *vs.* non-dimensional wavenumber Ka, for the submerged sphere with a three-cable power take-off system, with $\alpha = 60°$, $a/h = \frac{4}{5}$, for different values of tuned wavenumber $K_0 a$. ——, surge, $|\xi_1/A|$, – – –, heave, $|\xi_2/A|$.

increasing depth of submergence. However, in all cases the results remain valid for a considerable range of Ka, for given α, a/h and $K_0 a$. Values of $|\xi_j/A|$ were also calculated for the case $\alpha = 45°$ and showed similar trends.

From the above results it seems clear that a submerged sphere, with a three-cable mooring and power take-off system, can be a good absorber of wave power. That is, it can absorb power from a crest length of incident wave which is greater than the diameter of the sphere. This is a considerable improvement on the results obtained by Evans (1976) for a sphere heaving on the free surface and is due to two factors. Firstly, the submerged sphere moves in both surge and heave and power is extracted from both modes. This is in contrast to the surface sphere which was only allowed to heave. Thus the power absorbed is increased. Secondly, there are no buoyancy restoring forces acting on the sphere, thus enabling it to be tuned to longer waves resulting in an increase in the power absorbed. In the case of the heaving surface sphere the 'built-in' spring due to buoyancy restoring forces prevents it being tuned to longer waves (see Evans 1976, §7). The results also show that for some of the situations considered linear theory is not valid and nonlinear effects need to be included in the formulation. These effects could lead to a reduction in the power absorbed by the sphere.

It is important to note in a realistic situation power would probably be absorbed from an incident wave by an array of devices, rather than by a single device as considered here. In such a situation the interaction effects between the devices could affect the power absorbed by the array. However, the results presented in this paper show that gaps between neighbouring devices do not necessarily mean that power will 'leak' through them as each device may absorb power from a crest length of incident wave greater than its own dimensions. Finally it must be said that although the idea of power take-off and mooring through the system described in this paper is attractive, there may be engineering difficulties in actually realising such a system in practice.

(b) *Added-mass and damping coefficients for a submerged sphere*

7. Statement of problem and method of solution

In order to calculate the added-mass and damping coefficients for a submerged sphere it is necessary to solve the radiation problem for ϕ_j ($j = 1, 2, 3$). Due to the symmetry of the sphere it is only necessary to solve the radiation problems for a surging ($j = 1$) and a heaving ($j = 2$) sphere. The solution for a swaying sphere ($j = 3$) may be obtained by rotating the solution for a surging sphere through 90° about the vertical axis.

Define R, r, θ, ψ as follows

$$\left.\begin{aligned} R &= (x^2+z^2)^{\frac{1}{2}}, \quad r = (R^2+(y-h)^2)^{\frac{1}{2}},\\ \tan\theta &= R/(y-h), \quad \tan\psi = z/x. \end{aligned}\right\} \tag{7.1}$$

The radiation potential $\mathrm{Re}\{\phi_j e^{i\omega t}\}$ ($j = 1, 2$) satisfies Laplace's equation (2.1) in the fluid, the free surface condition (2.2), and (2.3). Equation (2.9) leads to, on use of (7.1)

$$\left.\frac{\partial\phi_1}{\partial r}\right|_{r=a} = \sin\theta\cos\psi = P_1^1(\cos\theta)\cos\psi, \tag{7.2}$$

$$\left.\frac{\partial\phi_2}{\partial r}\right|_{r=a} = \cos\theta = P_1(\cos\theta), \tag{7.3}$$

where, following Thorne (1953),

$$P_n^m(\cos\theta) = \sin^m\theta\,\frac{d^mP_n(\cos\theta)}{d(\cos\theta)^m}.$$

It is also necessary to impose the condition that there are only outgoing waves as $R\to\infty$, that is,

$$R^{\frac{1}{2}}\left\{\frac{\partial\phi_j}{\partial R}+iK\phi_j\right\}\to 0 \quad \text{as} \quad R\to\infty \quad \text{for} \quad j = 1,2. \tag{7.4}$$

The method of solution is that used by Ursell (1950) to solve the problem of the scattering of an incident wave by a submerged circular cylinder. He placed an infinite series of multipole potentials, of unknown strengths, at the centre of the cylinder. Then by satisfying the boundary conditions on the cylinder he obtained an infinite system of linear simultaneous equations for the strengths of the multipoles. The same method of solution was used by Ogilvie (1963) who generalised Ursell's results by considering an oscillating cylinder.

In this paper use is made of the three dimensional multipole potentials, obtained by Thorne (1953), to solve the radiation problems for the sphere. The general three-dimensional multipole potential given by Thorne (1953) is (for $m \leqslant n$; $m = 0, 1, \ldots$)

$$\Phi_{mn} = \left\{\frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{(-1)^{n+m-1}}{(n-m)!}\,\mathop{⨍}_0^\infty \frac{K+k}{K-k}\,k^n e^{-k(y+h)}J_m(kR)\,dk \right.$$
$$\left. -\,2\pi i\,\frac{(-1)^{m+n}}{(n-m)!}\,K^{n+1}e^{-K(y+h)}J_m(KR)\right\}\begin{matrix}\cos\\ \sin\end{matrix}\, m\psi, \tag{7.5}$$

where the bar across the integral sign denotes a principal value integral. For

$$r \leqslant 2h' < 2h,$$

ϕ_{mn} may be expanded as

$$\phi_{mn} = \left\{\frac{P_n^m(\cos\theta)}{r^{n+1}} + \sum_{s=m}^{\infty} (C_s - iD_s)\, r^s P_s^m(\cos\theta)\right\} \begin{matrix}\cos\\ \sin\end{matrix}\, m\psi, \tag{7.6}$$

where

$$C_s = \frac{(-1)^{n+s-1}}{(m+s)!\,(n-m)!}\int_0^\infty \frac{K+k}{K-k}\, k^{n+s} e^{-2kh}\,dk, \tag{7.7}$$

$$D_s = \frac{2\pi(-1)^{n+s}}{(m+s)!\,(n-m)!}\, K^{n+s+1} e^{-2Kh}. \tag{7.8}$$

As $R \to \infty$,

$$\phi_{mn} \sim \frac{2\pi i(-1)^{n+m-1}}{(n-m)!}\, K^{n+1} e^{-K(y+h)} H_m^{(2)}(KR) \begin{matrix}\cos\\ \sin\end{matrix}\, m\psi. \tag{7.9}$$

Here the multipole potential ϕ_{mn} has been written in complex form rather than the real form given by Thorne and his y, z, α become z, y, ψ, respectively, in the notation of this paper. The potential ϕ_{mn} satisfies Laplace's equation (2.1) in the fluid, the free surface condition (2.2), equation (2.3) and the radiation condition (7.4).

From the boundary condition on the sphere (7.2) it is clear that ϕ_1 need only be expressed as a sum of multipoles for $m = 1$, with a simple $\cos\psi$ dependence. Similarly, from (7.3), ϕ_2 may be expressed as a sum of multipoles for $m = 0$, independent of ψ. In §§8 and 9 below the solutions for ϕ_1 and ϕ_2 respectively are given.

8. The surging sphere

Write ϕ_1 as the following sum of multipole potentials with unknown strengths q_n,

$$\phi_1 = \sum_{n=1}^{\infty} \frac{q_n a^{n+2}}{(n+1)} \left\{\frac{P_n^1(\cos\theta)}{r^{n+1}} + \frac{(-1)^n}{(n-1)!}\int_0^\infty \frac{K+k}{K-k}\, k^n e^{-k(y+h)} J_1(kR)\,dk \right.$$
$$\left. - \frac{2\pi i(-1)^{n+1}}{(n-1)!}\, K^{n+1} e^{-K(y+h)} J_1(KR)\right\} \cos\psi. \tag{8.1}$$

From (7.6)–(7.8), ϕ_1 may be expanded, for $r \leqslant 2h' < 2h$, as

$$\phi_1 = \sum_{n=1}^{\infty} \frac{q_n a^{n+2}}{(n+1)} \left\{\frac{P_n^1(\cos\theta)}{r^{n+1}} + \sum_{s=1}^{\infty} B_{ns} r^s P_s^1(\cos\theta)\right\} \cos\psi, \tag{8.2}$$

where

$$B_{ns} = \frac{(-1)^{n+s-1}}{(n-1)!\,(s+1)!}\int_0^\infty \frac{K+k}{K-k}\, k^{n+s} e^{-2kh}\,dk - \frac{2\pi i(-1)^{s+n}}{(n-1)!\,(s+1)!}\, K^{n+s+1} e^{-2Kh}. \tag{8.3}$$

From (7.9) and (8.1), as $R \to \infty$

$$\phi_1 \sim A_1 e^{-Ky} H_1^{(2)}(KR) \cos\psi, \tag{8.4}$$

where

$$A_1 = -2\pi i a e^{-Kh} \sum_{n=1}^{\infty} \frac{(Ka)^{n+1}(-1)^{n+1}}{(n+1)\,(n-1)!}\, q_n. \tag{8.5}$$

To determine the q_n differentiate (8.2) with respect to r and apply the boundary condition (7.2) to obtain

$$P_1^1(\cos\theta)\cos\psi = \sum_{n=1}^{\infty} q_n \left\{-P_n^1(\cos\theta) + \sum_{s=1}^{\infty} B_{ns} \frac{s a^{n+s+1}}{(n+1)}\, P_s^1(\cos\theta)\right\} \cos\psi.$$

As the $P_m^1(\cos\theta)$ are orthogonal equate coefficients of $P_m^1(\cos\theta)$ to obtain

$$\delta_{m1} = -q_m + \sum_{n=1}^{\infty} B_{nm}\frac{ma^{m+n+1}}{(n+1)}q_n \quad \text{for} \quad m = 1, 2, \ldots. \tag{8.6}$$

This is an infinite system of linear simultaneous equations for an infinite number of unknowns. It can be solved numerically by truncating the system to a finite number of terms; this method of solution is discussed in §10.

In general the added-mass and damping coefficients, denoted by a_{ij} and b_{ij} respectively, are given by

$$\omega^2 a_{ij} - i\omega b_{ij} = -\rho\omega^2\int_S \phi_j \frac{\partial\phi_i}{\partial n}dS \quad \text{for} \quad i,j = 1, \ldots, 6.$$

For the submerged sphere, $i,j = 1, 2, 3$ and $a_{ij} = 0$, $b_{ij} = 0$ for $i \neq j$ by symmetry. Also $a_{33} = a_{11}$ and $b_{33} = b_{11}$ again by symmetry. Now

$$\omega^2 a_{11} - i\omega b_{11} = -\rho\omega^2\int_0^{2\pi}\int_0^{\pi}\phi_1|_{r=a}\cdot\left.\frac{\partial\phi_1}{\partial r}\right|_{r=a} a^2\sin\theta\,d\theta\,d\psi,$$

$$= -\rho\omega^2 a^2\int_0^{2\pi}\int_0^{\pi}\phi_1|_{r=a}P_1^1(\cos\theta)\sin\theta\cos\psi\,d\theta\,d\psi,$$

on use of (7.2). From (8.2) and (8.6) ϕ_1 may be written as

$$\phi_1 = a\cos\psi\left\{\sum_{n=1}^{\infty} q_n\left[\frac{1}{(n+1)}\left(\frac{a}{r}\right)^{n+1} + \frac{1}{n}\left(\frac{r}{a}\right)^n\right]P_n^1(\cos\theta) + \left(\frac{r}{a}\right)P_1^1(\cos\theta)\right\}.$$

Thus

$$\omega^2 a_{11} - i\omega b_{11} = -\rho\omega^2 a^2\int_0^{2\pi}\cos^2\psi\,d\psi\,.\,[aq_1.\ +a]\,.\,\tfrac{4}{3},$$

$$= -[\tfrac{4}{3}\pi a^3\rho\omega^2]\,.\,(\tfrac{3}{2}q_1 + 1), \tag{8.7}$$

using

$$\int_0^{\pi}[P_n^1(\cos\theta)]^2\sin\theta\,d\theta = 2n(n+1)(2n+1)^{-1},$$

and

$$\int_0^{\pi}P_n^1(\cos\theta)P_m^1(\cos\theta)\sin\theta\,d\theta = 0 \quad \text{for} \quad m \neq n.$$

By an application of Green's theorem it is possible to show that (cf. Newman, 1976, equation (31*b*))

$$b_{11} = \rho\omega K^{-1}|A_1|^2. \tag{8.8}$$

This relation can be used to check the consistency of the numerical results obtained from solving (8.6).

9. The heaving sphere

The analysis follows that of §8. Write ϕ_2 as

$$\phi_2 = \sum_{n=1}^{\infty}\frac{p_n a^{n+2}}{(n+1)}\left\{\frac{P_n(\cos\theta)}{r^{n+1}} + \frac{(-1)^{n-1}}{n!}\int_0^{\infty}\frac{K+k}{K-k}k^n e^{-k(y+h)}J_0(kR)\,dk - \frac{2\pi i(-1)^n}{n!}K^{n+1}e^{-K(y+h)}J_0(KR)\right\}. \tag{9.1}$$

Note that it is possible to take the summation from $n = 0$ to ∞ and so include the multipole potential with a r^{-1} singularity at the centre of the sphere. However, when the boundary condition (7.3) is applied it is found that the coefficient p_0 is zero. Physically this corresponds to requiring that, at any instant of time, there is no flux of fluid across the boundary of the sphere because it is a rigid body. If the sphere was pulsating say, that is if it were not a rigid body, then the r^{-1} singularity would be required to obtain a solution. This problem does not arise for a surging sphere as there the solution depends on $P_n^1(\cos\theta)$ and n must be greater than or equal to 1, thus excluding the r^{-1} singularity. For $r \leqslant 2h' < 2h$,

$$\phi_2 = \sum_{n=1}^{\infty} \frac{p_n a^{n+2}}{(n+1)} \left\{ \frac{P_n(\cos\theta)}{r^{n+1}} + \sum_{s=0}^{\infty} A_{ns} r^s P_s(\cos\theta) \right\}, \tag{9.2}$$

where

$$A_{ns} = \frac{(-1)^{n+s-1}}{n!\,s!} \int_0^\infty \frac{K+k}{K-k} k^{n+s} e^{-2kh} dk - \frac{2\pi i(-1)^{n+s}}{n!\,s!} K^{n+s+1} e^{-2Kh}. \tag{9.3}$$

As $R \to \infty$

$$\phi_2 \sim \tfrac{1}{2} A_2 e^{-Kz} H_0^{(2)}(KR), \tag{9.4}$$

where

$$A_2 = 4\pi i a e^{-Kh} \sum_{n=1}^{\infty} \frac{(Ka)^{n+1}(-1)^{n+1}}{(n+1)!} p_n. \tag{9.5}$$

Differentiating ϕ_2 with respect to r and using boundary conditions (7.3) gives

$$\delta_{m1} = -p_m + \sum_{n=1}^{\infty} A_{nm} \frac{m a^{m+n+1}}{(n+1)} p_n \quad \text{for} \quad m = 1, 2, \ldots. \tag{9.6}$$

The added-mass and damping coefficients for the heaving submerged sphere are given by

$$\begin{aligned} \omega^2 a_{22} - i\omega b_{22} &= -\rho\omega^2 a^2 \int_0^{2\pi} \int_0^\pi \phi_2|_{r=a} P_1(\cos\theta) \sin\theta \, d\theta \, d\psi, \\ &= -[\tfrac{4}{3}\pi a^3 \rho\omega^2].(\tfrac{3}{2}p_1 + 1), \end{aligned} \tag{9.7}$$

where use has been made of (9.2), (9.6) and

$$\int_0^\pi [P_n(\cos\theta)]^2 \sin\theta \, d\theta = 2(2n+1)^{-1},$$

$$\int_0^\pi P_n(\cos\theta) P_m(\cos\theta) \sin\theta \, d\theta = 0 \quad \text{for} \quad m \neq n.$$

An application of Green's theorem gives

$$b_{22} = \tfrac{1}{2}\rho\omega K^{-1} |A_2|^2. \tag{9.8}$$

10. Solution of the systems of linear simultaneous equations

From the results given in §§8 and 9 it can be seen that all that is required for a full solution to the problem of a surging, heaving or swaying sphere are the coefficients p_n, q_n. These may be determined from a numerical solution of the systems of linear simultaneous equations given by (8.6) and (9.6). In order to simplify the numerical work involved the systems of equations (8.6) and (9.6) may be decomposed into equivalent real systems. In this way the real and imaginary parts p_n, q_n may be more easily calculated. This is done in the following manner.

Write p_n, q_n as

$$\left.\begin{aligned} p_n &= \left\{\frac{(n+1)}{(-Ka)^{n+1}}\right\}(a_n+ib_n), \\ q_n &= \left\{\frac{(n+1)}{(-Ka)^{n+1}}\right\}(c_n+id_n), \end{aligned}\right\} \tag{10.1}$$

where a_n, b_n, c_n, d_n are all real. Thus by taking the real and imaginary parts of (8.6), (9.6) and using (10.1)

$$\left.\begin{aligned} -\delta_{m1}\frac{(-Ka)^{m+1}}{(m+1)} &= a_m + \sum_{n=1}^{\infty}\frac{m}{m+1}\left\{a_n\alpha_{nm} - \frac{2\pi e^{-Kh}(Ka)^{2m+1}}{m!\,n!}b_n\right\}, \\ 0 &= b_m + \sum_{n=1}^{\infty}\frac{m}{m+1}\left\{b_n\alpha_{nm} + \frac{2\pi e^{-Kh}(Ka)^{2m+1}}{m!\,n!}a_n\right\}, \\ -\delta_{m1}\frac{(-Ka)^{m+1}}{(m+1)} &= c_m + \sum_{n=1}^{\infty}\frac{m}{m+1}\left\{c_n\beta_{nm} - \frac{2\pi e^{-Kh}(Ka)^{2m+1}}{(n-1)!\,(m+1)!}d_n\right\}, \\ 0 &= d_m + \sum_{n=1}^{\infty}\frac{m}{m+1}\left\{d_n\beta_{nm} + \frac{2\pi e^{-Kh}(Ka)^{2m+1}}{(n-1)!\,(m+1)!}c_n\right\}, \end{aligned}\right\} \tag{10.2}$$

where

$$\left.\begin{aligned} \alpha_{nm} &= \frac{(Ka)^{2m+1}}{m!}\frac{(m+n)!}{n!}F_{m+n}, \\ \beta_{nm} &= \frac{(Ka)^{2m+1}}{(m+1)!}\frac{(m+n)!}{(n-1)!}F_{m+n}, \\ F_s &= \frac{2}{s!}\left\{e^{-2Kh}Ei(2Kh) - \sum_{j=1}^{s+1}(j-1)!\,(2Kh)^{-j}\right\} + (2Kh)^{-(s+1)}, \\ Ei(x) &= \int_{-\infty}^{x} t^{-1}e^{t}dt. \end{aligned}\right\} \tag{10.3}$$

Now define S_x, Z_x, ϵ_m, γ_m, η_m, σ_m (all real) to be such that

$$S_x = 2\pi e^{-Kh}\sum_{n=1}^{\infty}\frac{x_n}{n!}, \quad Z_x = 2\pi e^{-Kh}\sum_{n=1}^{\infty}\frac{x_n}{(n-1)!}, \tag{10.4}$$

$$\left.\begin{aligned} \epsilon_m + \frac{m}{m+1}\sum_{n=1}^{\infty}\alpha_{nm}\epsilon_n &= \left(\frac{m}{m+1}\right)\frac{(Ka)^{2m+1}}{m!}, \\ \gamma_m + \frac{m}{m+1}\sum_{n=1}^{\infty}\alpha_{nm}\gamma_n &= \delta_{m1}, \end{aligned}\right\} \tag{10.5}$$

$$\left.\begin{aligned} \sigma_m + \frac{m}{m+1}\sum_{n=1}^{\infty}\beta_{nm}\sigma_n &= \left(\frac{m}{m+1}\right)\frac{(Ka)^{2m+1}}{(m+1)!}, \\ \eta_m + \frac{m}{m+1}\sum_{n=1}^{\infty}\beta_{nm}\eta_n &= \delta_{m1}. \end{aligned}\right\} \tag{10.6}$$

Now using (10.2)–(10.6) it is possible to express a_m, b_m, c_m, d_m

$$\left.\begin{aligned} a_m &= S_b\epsilon_m - \tfrac{1}{2}(Ka)^2\gamma_m, \quad b_m = -S_a\epsilon_m, \\ c_m &= Z_d\sigma_m - \tfrac{1}{2}(Ka)^2\eta_m, \quad d_m = -Z_c\sigma_m. \end{aligned}\right\} \tag{10.7}$$

Furthermore

$$\left.\begin{aligned} S_a &= -\tfrac{1}{2}(Ka)^2 S_\gamma (1+S_\epsilon^2)^{-1}, \\ S_b &= \tfrac{1}{2}(Ka)^2 S_\gamma S_\epsilon (1+S_\epsilon^2)^{-1}, \\ Z_c &= -\tfrac{1}{2}(Ka)^2 Z_\eta (1+Z_\sigma^2)^{-1}, \\ Z_d &= \tfrac{1}{2}(Ka)^2 Z_\eta Z_\sigma (1+Z_\sigma^2)^{-1}. \end{aligned}\right\} \tag{10.8}$$

Using the above results equations (8.5), (9.5) may be written as

$$\left.\begin{aligned} A_1/a &= \tfrac{1}{2}e^{Kh}(Ka)^2 Z_\eta (Z_\sigma + i)(1+Z_\sigma^2)^{-1}, \\ A_2/a &= -e^{Kh}(Ka)^2 S_\gamma (S_\epsilon + i)(1+S_\epsilon^2)^{-1}, \end{aligned}\right\} \tag{10.9}$$

also, equations (8.7), (9.7) give

$$\left.\begin{aligned} \mu_{11} &= \tfrac{3}{2}\eta_1 - 1 - \tfrac{3}{2}Z_\eta Z_\sigma \sigma_1 (1+Z_\sigma^2)^{-1}, \\ \lambda_{11} &= \tfrac{3}{2}Z_\eta \sigma_1 (1+Z_\sigma^2)^{-1}, \end{aligned}\right\} \tag{10.10}$$

$$\left.\begin{aligned} \mu_{22} &= \tfrac{3}{2}\gamma_1 - 1 - \tfrac{3}{2}S_\gamma S_\epsilon \epsilon_1 (1+S_\epsilon^2)^{-1}, \\ \lambda_{22} &= \tfrac{3}{2}S_\gamma \epsilon_1 (1+S_\epsilon^2)^{-1}, \end{aligned}\right\} \tag{10.11}$$

where μ_{jj} and λ_{jj} are the non-dimensionalized added-mass and damping coefficients given by

$$\mu_{jj} = a_{jj}(\tfrac{4}{3}\pi a^3 \rho)^{-1}, \quad \lambda_{jj} = b_{jj}(\tfrac{4}{3}\pi a^3 \rho\omega)^{-1}, \quad \text{for} \quad j = 1, 2, 3. \tag{10.12}$$

In the above, the two complex systems (8.6) and (9.6) have been reduced to four real systems (10.5), (10.6) which are easier to evaluate numerically on a computer. Furthermore it has been shown how the added mass and damping coefficients and the amplitudes of the waves at $R = \infty$ can be expressed in terms of the solutions to the real systems (10.5), (10.6).

The four systems (10.5), (10.6) were solved numerically by truncating the infinite system to a finite number of equations and unknowns. The finite systems were then solved by using a standard computer procedure for the numerical solution of a system of real, linear, simultaneous equations. Solutions were computed successively with 10, 15, 20 terms taken into account. The results for the added-mass and damping coefficients and the wave amplitudes were found to agree to four or five significant figures in general. It was found that for $Ka < 0{\cdot}1$ and $a/h \geqslant 0{\cdot}8$ agreement was less good; though by decreasing a/h the agreement between solutions with 10, 15, 20 terms was good for Ka down to 0.05. The relations (8.8), (9.8) were also used as a check on the calculations, by comparing results from (10.10) and (10.11) for λ_{11}, and λ_{22} with those obtained by using (8.8), (9.8), (10.9) and (10.12). Again agreement was found to four or five significant figures. The results obtained for the added-mass and damping coefficients are presented and discussed below.

11. Results and discussion for the added-mass and damping coefficients of the sphere

Using the results of §10 calculations were carried out for μ_{jj}, λ_{jj} $(j = 1, 2)$ for the following range of parameters

$$a/h = \tfrac{4}{5}, \tfrac{2}{3}, \tfrac{4}{7}, \tfrac{1}{2}, \tfrac{1}{3} \quad 0{\cdot}1 < Ka < 2,$$

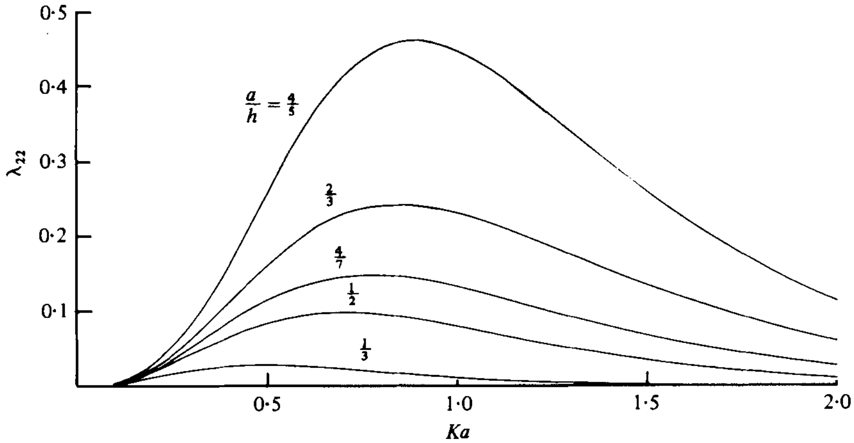

FIGURE 13. Non-dimensional damping coefficient λ_{22} *vs.* non-dimensional wavenumber Ka, for a heaving submerged sphere, for different values of a/h.

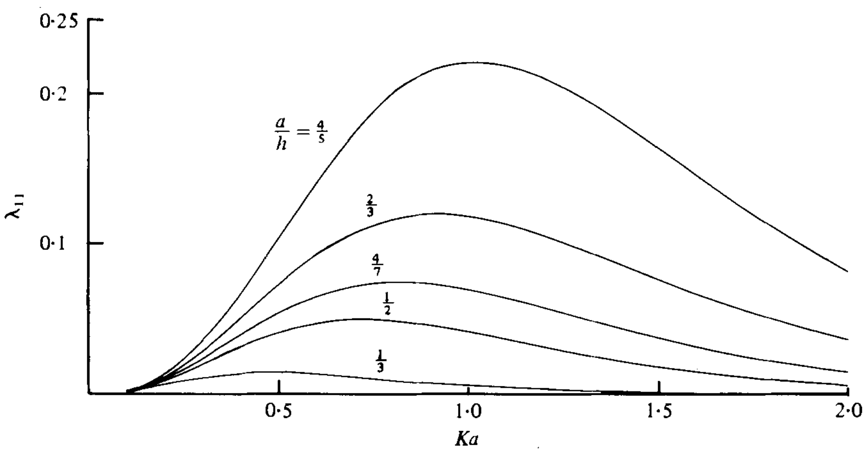

FIGURE 14. Non-dimensional damping coefficient λ_{11} *vs.* non-dimensional wavenumber Ka, for a surging submerged sphere, for different values of a/h.

where a/h is the non-dimensional submergence parameter and Ka is the non-dimensional wavenumber. Note that $a/h < 1$ and for a deeply submerged sphere $a/h \to 0$. Curves of μ_{jj}, λ_{jj} are given in figures 13–16.

Figures 13 and 14 show λ_{22} and λ_{11}, respectively, plotted against Ka for varying values of a/h. It can be seen that as a/h decreases, that is, as the sphere becomes more deeply submerged, the damping coefficient also decreases. This is not surprising as the damping coefficient and the waves radiated to infinity are related [see equations (8.8) and (9.8)]. So if the depth of submergence of the sphere is increased it is to be expected that the amplitude of the waves generated will decrease and thus the damping coefficient will decrease. Note that there is a factor of two difference in the scales marked on figures 13 and 14, showing that the damping coefficient for a surging sphere is generally smaller than that for a heaving sphere.

In figures 15 and 16 curves are given for μ_{22} and μ_{11}, respectively. Here as the depth

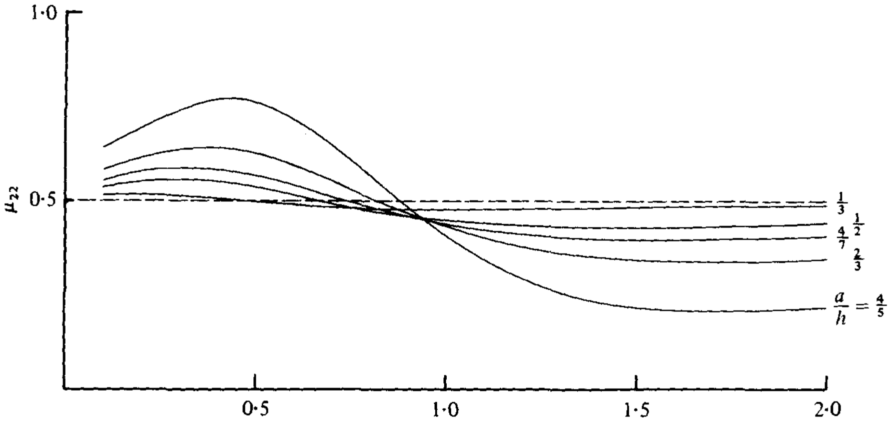

FIGURE 15. Non-dimensional added-mass coefficient μ_{22} *vs.* non-dimensional wavenumber Ka, for a heaving submerged sphere, for different values of a/h.

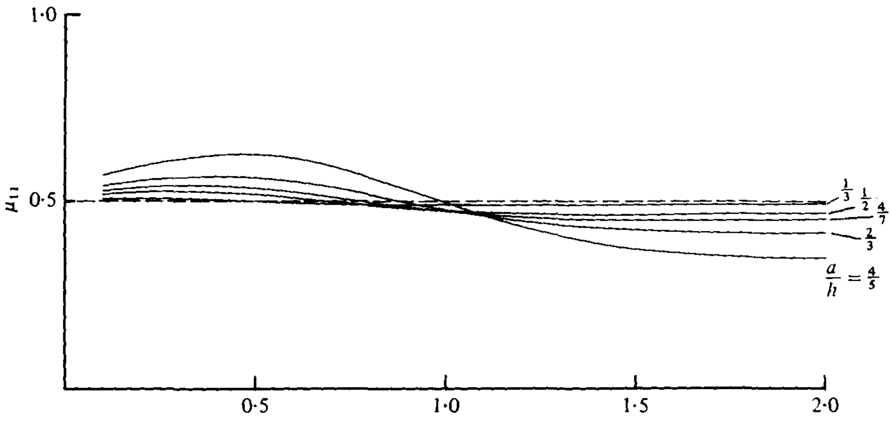

FIGURE 16. Non-dimensional added-mass coefficient μ_{11} *vs.* non-dimensional wavenumber Ka, for a surging submerged sphere, for different values of a/h.

of submergence is increased ($a/h \to 0$) both μ_{22} and μ_{11} approach the value of a half. This is to be expected as the non-dimensional added-mass of a sphere oscillating in an infinite fluid with no free surface take the value one half (see Newman 1977, p. 144, equation 134). As the sphere becomes deeply submerged the effect of the free surface is minimal, and so the sphere can be considered to be moving in an infinite fluid. Note that when the sphere is closer to the free surface μ_{22} deviates more from the value of one half than μ_{11}.

If the case of a sphere oscillating in an infinite fluid, with no free surface, is considered as the limiting case of the submerged sphere, when its depth of submergence becomes large, then the following explanation of the difference between the results for heave and surge is plausible. When the submerged sphere is heaving its highest point is forced nearer the undisturbed free surface than when it is surging. In surge the sphere's motion is parallel to the undisturbed free surface and the highest point of the sphere is always the same distance from it. So it may be argued that the heaving sphere 'interacts' more strongly with the free surface than the surging sphere. Thus a larger deviation from the results for limiting case of a sphere oscillating in an infinite fluid occurs when the sphere heaves. (Note that the damping coefficient for a sphere in an infinite fluid, with no free surface, is zero.)

Finally, as stated in §8, the results for the non-dimensional added-mass and damping of a swaying submerged sphere are identical to those for a surging submerged sphere, by symmetry. Thus λ_{33} and μ_{33} are the same as λ_{11} and μ_{11} in figures 14 and 16, respectively.

12. Conclusion

In part (*a*) of this paper a linearized theory has been presented for the absorption of wave power by a submerged sphere. This analysis included the effects of a possible mooring and power take-off system for the sphere. The differences between a single-cable and a three-cable system have been examined. It has been shown that the three-cable system allows the sphere to absorb power from a crest length of incident wave greater than its diameter. For a single-cable system it does not seem possible to achieve this result. Curves of the absorption length for both mooring and power take-off systems have been given, showing how the absorption length varies with changes of tuned wavenumber, depth of submergence and the wavenumber of the incident wave. These results suggest that the submerged sphere could be a good wave-power absorber for certain values of the parameters. However, the results also show that, for other values of the parameters, the linearized theory used is inappropriate and a nonlinear theory is necessary to describe the motion of the sphere. Finally, part (*a*) shows that the submerged sphere compares favourably with the floating sphere (see Evans 1976) as a wave-power absorber.

Part (*b*) of the paper concentrated on the solution of the radiation problems, for the forced motion of a sphere in heave or surge. The results for the added-mass and damping coefficients were derived and used in part (*a*) to calculate the absorption length and other properties of the sphere acting as an absorber of wave power. The method of solution used, that is the multipole potential method, is only applicable for certain body shapes and is not generally useful in solving similar problems. However, in the case of a submerged sphere it is probably the most straightforward method of solution.

It is hoped to extend these results to the case of two or more adjacent spheres acting as wave-power absorbers. This would obviously be a better representation of how wave power absorbers are likely to be moored at sea. That is, not individual devices spaced out at large distance, and thus effectively independent in their operation, but arrays of devices in reasonably close proximity. In such arrays the motion of one device could significantly affect the wave-power absorption properties of another device nearby.

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